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Fixed Point Theorem on Compatible Mappings of Type (E)

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ABSTRACT: The purpose of this paper is to present a common fixed point theorem in a metric space which extends the result of Bijendra Singh and M.S. Chauhan using the weaker conditions such as compatible mappings of type (E) and associated sequence in place of compatibility and completeness of the metric.

Keywords: Fixed point, self maps, compatible mappings, weakly compatible mappings, compatible mappings of type (E) and associated sequence.

AMS (2000) Mathematics Classification: 54H25,47H10

I. INTRODUCTION

Fixed point theory is an important area of functional analysis. The study of common fixed point of mappings satisfying contractive type condition has been a very active field of research. G. Jungck [1] introduced the concept of compatible maps which is weaker than weakly commuting maps. After wards Jungck and Rhoades [4] defined weaker class of maps known as weakly compatible maps. This concept has been frequently used to prove existence theorem in common fixed point theory.

M.R. Singh and Y.M. Singh [6] introduced the concept of compatible mappings of type (E).In this paper we prove a common fixed point theorem for four self maps in which one pair is compatible mapping of type (E) and other pair is weakly compatible.

II. DEFINITIONS AND PRELIMINARIES

2.1 Compatible mappings

Two self maps S and T of a metric space (X,d) are said to be compatible mappings if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever $< x_n >$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

2.2 Weakly compatible

Two self maps S and T of a metric space (X,d) are said to be weakly compatible if they commute at their coincidence point. i.e. if $Su = Tu$ for some $u \in X$ then $STu = TSu$.

2.3 Compatible mappings of type (E)

Two self maps S and T of metric space (X,d) are said to be compatible mappings of type (E) if $\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} STx_n = Tt$ and $\lim_{n\to\infty} TTx_n = \lim_{n\to\infty} TSx_n = St$, whenever $< x_n >$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Bijenrda Singh and M.S. Chauhan [5] proved the following theorem.

2.4 Theorem: Let A,B,S and T be self mappings from a complete metric space (X,d) into itself satisfying the following conditions

(2.4.1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$

(2.4.2) one of A,B,S and T is continuous

(2.4.3) $[d(Ax,By)]^2 \le k_1[d(Ax,Sx)d(By,Ty)+d(By,Sx)d(Ax,Ty)]$

$+k_2[d(Ax,Sx)d(Ax,Ty)+d(By,Ty)d(By,Sx)]$ where $0 \le k_1+2k_2 < 1, k_1,k_2 \ge 0$

(2.4.4) the pairs (A,S) and (B,T) are compatible on X

further, if X is a complete metric space then A,B,S and T have a unique common fixed point in X.

Now we generalize the theorem using compatible mappings of type (E) and associated sequence.

2.5 Associated sequence [7]: Suppose A, B, S and T are self maps of a metric space (X,d) satisfying the condition (2.5.1).Then for an arbitrary $x_0 \in X$ such that $Ax_0 = Tx_1$ and for this point x_1, there exist a point x_2 in X such that $Bx_1 = Sx_2$ and so on. Proceeding in the similar manner, we can define a sequence $< x_n >$ in X such that $y_{2n} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+1} = Bx_{2n+2} = Sx_{2n+1}$ for $n \ge 0$. We shall call this sequence as an "Associated sequence of x_0" relative to four self maps A, B, S and T.

Now we prove a lemma which plays an important role in our main theorem.

2.6 Lemma: Let A, B, S and T be self mappings from a complete metric space (X, d) into itself satisfying the conditions (2.4.1) and (2.4.3). Then the associated sequence $\{y_n\}$ relative to four self maps is a Cauchy sequence in X.

Proof: From the conditions (2.4.1), (2.4.3) and from the definition of associated sequence, we have

$$\begin{aligned}
\left[d(y_{2n+1},y_{2n})\right]^2 &= \left[d(Ax_{2n},Bx_{2n-1})\right]^2 \\
&\le k_1\left[d(Ax_{2n},Sx_{2n})\ d(Bx_{2n-1},Tx_{2n-1})+d(Bx_{2n-1},Sx_{2n})\ d(Ax_{2n},Tx_{2n-1})\right] \\
&\quad + k_2\left[d(Ax_{2n},Sx_{2n})\ d(Ax_{2n},Tx_{2n-1})+d(Bx_{2n-1},Tx_{2n-1})\ d(Bx_{2n-1}\ Sx_{2n})\right] \\
&= k_1\left[d(y_{2n+1},y_{2n})\ d(y_{2n},y_{2n-1})+0\right] \\
&\quad + k_2\left[d(y_{2n+1},y_{2n})\ d(y_{2n+1},y_{2n-1})+0\right]
\end{aligned}$$

This implies

$$d(y_{2n+1},y_{2n}) \le k_1\ d(y_{2n},y_{2n-1}) + k_2\left[d(y_{2n+1},y_{2n}) + d(y_{2n},y_{2n-1})\right]$$

$$d(y_{2n+1},y_{2n}) \le h\ d(y_{2n},y_{2n-1})$$

$$where\ h = \frac{k_1+k_2}{1-k_2} < 1$$

for every integer $p > 0$, *we get*

$$\begin{aligned}
d(y_n,y_{n+p}) &\le d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+\dots\dots\dots\dots+ d(y_{n+p-1},y_{n+p}) \\
&\le h^n d(y_0,y_1)+h^{n+1}d(y_0,y_1)+\dots\dots\dots\dots+h^{n+p-1}d(y_0,y_1) \\
&\le \left(h^n+h^{n+1}+\dots\dots\dots\dots+h^{n+p-1}\right)d(y_0,y_1) \\
&\le h^n\left(1+h+h^2+\dots\dots\dots\dots+h^{p-1}\right)d(y_0,y_1)
\end{aligned}$$

Since h<1, $h^n \to 0$ as $n \to \infty$, so that $d(y_n, y_{n+p}) \to 0$. This shows that the sequence $\{y_n\}$ is a Cauchy sequence in X and since X is a complete metric space; it converges to a limit, say $z \in X$.

The converse of the Lemma is not true, that is A,B,S and T are self maps of a metric space (X,d) satisfying (2.4.1) and (2.4.3), even if for $x_0 \in X$ and for associated sequence of x_0 converges, the metric space (X,d) need not be complete.
The following example establishes this.

2.7 Example: Let $X = (0,1]$ with $d(x,y) = |x-y|$. Define self maps of A, B, S and T of X by

$$Ax = \begin{cases} \frac{1-x}{2} & if\ x \in \left(0,\frac{1}{2}\right) - \left\{\frac{1}{4}\right\} \\ \frac{1}{8} & if\ x = \frac{1}{4} \\ \frac{3x-1}{2} & if\ x \in \left[\frac{1}{2},1\right] \end{cases} \qquad Bx = \begin{cases} \frac{1}{4} & if\ x \in \left(0,\frac{1}{2}\right) - \left\{\frac{1}{4}\right\} \\ \frac{1}{8} & if\ x = \frac{1}{4} \\ x & if\ x \in \left[\frac{1}{2},1\right] \end{cases}$$

$$Sx = \begin{cases} \frac{x}{2} & if\ x \in \left(0,\frac{1}{2}\right) - \left\{\frac{1}{4}\right\} \\ \frac{3}{8} & if\ x = \frac{1}{4} \\ x & if\ x \in \left[\frac{1}{2},1\right] \end{cases} \qquad Tx = \begin{cases} x & if\ x \in \left(0,\frac{1}{2}\right) - \left\{\frac{1}{4}\right\} \\ \frac{1}{8} & if\ x = \frac{1}{4} \\ x^2 & if\ x \in \left[\frac{1}{2},1\right] \end{cases}$$

Then $A(X) = \left(\frac{1}{4},\frac{1}{2}\right) \cup \left\{\frac{1}{8}\right\} \cup \left[\frac{1}{4},1\right]$, $B(X) = \left\{\frac{1}{4}\right\} \cup \left\{\frac{1}{8}\right\} \cup \left[\frac{1}{2},1\right]$, $Sx = \left(0,\frac{1}{4}\right) \cup \left\{\frac{3}{8}\right\} \cup \left[\frac{1}{2},1\right]$ and $T(X) = \left(0,\frac{1}{2}\right) \cup \left\{\frac{1}{8}\right\} \cup \left[\frac{1}{4},1\right]$ so that the conditions $A(X) \subset T(X)$ and $B(X) \subset S(X)$ are satisfied.

The associated sequence $Ax_0, Bx_1, Ax_2, Bx_3....Ax_{2n}, Bx_{2n+1},....$ converges to the point '1', but X is not a complete metric space.

III. MAIN RESULT

3.1 Theorem: Let A, B, S and T be self mappings from a metric space (X,d) into itself satisfying the following conditions

$A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ **...** (3.1.1)

$$[d(Ax,By)]^2 \leq k_1[d(Ax,Sx)d(By,Ty) + d(By,Sx)d(Ax,Ty)] + k_2[d(Ax,Sx)d(Ax,Ty) + d(By,Ty)d(By,Sx)] \quad \text{... (3.1.2)}$$

for all x,y in X where $0 \leq k_1 + 2k_2 < 1, k_1, k_2 \geq 0$

one of A,B,S and T is continuous ... (3.1.3)

the pair (A,S) is compatible mappings of type(E) and (B,T) is weakly compatible ... (3.1.4)

the sequence $Ax_0, Bx_1, Ax_2, Bx_{3,}...., Ax_{2n}, Bx_{2n+1}...$ converges to $z \in X$. ... (3.1.5)

Then A,B,S and T have a unique common fixed point in z in X.

Proof: From the condition (3.1.5), we have

$Ax_{2n} \to z, Tx_{2n+1} \to z, Bx_{2n+1} \to z\ and\ Sx_{2n} \to z$ as $n \to \infty$. ... (3.1.6)

Suppose A is continuous. Then $AAx_{2n} \to Az, ASx_{2n} \to Az$ as $n \to \infty$.

Since (A,S) is compatible mappings of type (E), then $AAx_{2n}, ASx_{2n} \to Sz$ and $SSx_{2n}, SAx_{2n} \to Az$ as $n \to \infty$. ... (3.1.7)

From the conditions (3.1.6) and (3.1.7), we have $Az = Sz$.
Since $A(X) \subseteq T(X)$ implies that there exists $u \in X$ such that $z = Tu$.

To prove $Tu = Bu$, put $x = x_{2n}, y = u$ in condition (3.1.2), we have

$$[d(Ax_{2n}, Bu)]^2 \le k_1[d(Ax_{2n}, Sx_{2n})d(Bu,Tu) + d(Bu, Sx_{2n})d(Ax_{2n},Tu)] + k_2[d(Ax_{2n}, Sx_{2n})d(Ax_{2n},Tu) + d(Bu,Tu)d(Bu, Sx_{2n})]$$

Letting $n \to \infty$ and using the conditions $Tu = z$ and $Az = Sz$, (3.1.6), then we get

$$[d(z, Bu)]^2 \le k_1[d(z,z)d(Bu,Tu) + d(Bu,z)d(z,z)] + k_2[d(z,z)d(z,z) + d(Bu,z)d(Bu,z)]$$

$$[d(z, Bu)]^2 \le k_2[d(Bu,z)]^2$$

$(1-k_2)[d(z,Bu)]^2 \le 0$, since $0 \le k_1 + k_2 < 1$

$d(z, Bu) = 0$ implies that $Bu = z$.

Hence $Bu = Tu = z$.

Since the pair (B,T) is weakly compatible, then $Bu = Tu$ implies $BTu = TBu$ .This gives $Bz = Tz$.
To prove $Az = z$, Put $x = z, y = u$ in condition (3.1.2), we have

$$[d(Az, Bu)]^2 \le k_1[d(Az, Sz)d(Bu,Tu) + d(Bu, Sz)d(Az,Tu)] + k_2[d(Az, Sz)d(Az,Tu) + d(Bu,Tu)d(Bu, Sz)]$$

Using conditions $Bu = Tu = z, Bz = Tz$ and $Az = Sz$, we get

$$[d(Az,z)]^2 \le k_1[d(Az,Az)d(Tu,Tu) + d(z,Az)d(Az,z)] + k_2[d(Az,Az)d(Az,z) + d(z,z)d(z,Sz)]$$

$$[d(Az,z)]^2 \le k_1[d(Az,z)]^2$$

$(1-k_1)[d(Az,z)]^2 \le 0$, since $0 \le k_1 + k_2 < 1$

$d(Az, z) = 0$ implies $Az = z$.

Therefore $Az = Sz = z$.
Hence z is a common fixed point of A and S.
Now put $x = z, y = z$ in condition (3.1.2), we have

$$[d(Az, Bz)]^2 \le k_1[d(Az, Sz)d(Bz,Tz) + d(Bz, Sz)d(Az,Tz)] + k_2[d(Az, Sz)d(Az,Tz) + d(Bz,Tz)d(Bz, Sz)]$$

Using the conditions $Az = Sz$ and $Bz = Tz$, we have

$$[d(z,Bz)]^2 \le k_1[d(z,z)d(Bz,Bz) + d(Bz,z)d(z,Bz)] + k_2[d(z,z)d(z,Tz) + d(Bz,Bz)d(Bz,z)]$$

$$[d(z,Bz)]^2 \le k_1[d(Bz,z)]^2$$

$(1-k_1)[d(z,Bz)]^2 \le 0$, since $0 \le k_1 + k_2 < 1$

$d(Bz, z) = 0$ implies $Bz = z$.

Hence $Bz = Tz = z$.

Since $Bz = Tz = Az = Sz = z$, we get z is a common fixed point of A, B, S and T. The uniqueness of the fixed point can be easily proved.

3.2 Remark: From the example given above, clearly the pair (A, S) is compatible mappings of type (E) and the pair (B, T) is weakly compatible as they commute at coincident point 1.
But the pair (A,S) is not at all compatible, compatible mappings of type(A), compatible mappings of type(B), compatible mappings of type(P).

For this, take a sequence $x_n = \frac{1}{2} - \frac{1}{n}$, for $n \geq 1$, then $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \frac{1}{4} = t$ (Say), $\lim_{n\to\infty} AAx_n = \lim_{n\to\infty} ASx_n = S(t) = \frac{3}{8}$ and $\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} SAx_n = A(t) = \frac{1}{8}$ .Also the condition (3.1.2) holds for the values of $0 \leq k_1 + 2k_2 < 1$, where $k_1, k_2 \geq 0$. We note that X is not a complete metric space and it is easy to prove that the associated sequence $Ax_0, Bx_1, Ax_2, Bx_3....Ax_{2n}, Bx_{2n+1},....$ converges to the point 1 which is a common fixed point of A, B, S and T. In fact '1' is the unique common fixed point of A, B, S and T.

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